Connected Sets

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Note: In this document, a juxtaposition of two points denotes the distance between them.

1 Kinds of connectedness

Unsplittable. Two point sets T_1 and T_2 **split** a point set S if the following conditions all hold:

- T_1 and T_2 are disjoint open sets.
- T_1 and T_2 both intersect S ($T_1 \cap S \neq \emptyset$ and $T_2 \cap S \neq \emptyset$).
- T_1 and T_2 together cover S ($S \subseteq T_1 \cup T_2$).

A point set S is **unsplittable** if there do not exist sets T_1 and T_2 that split it. According to several sources, when mathematicians say "connected", they usually mean this definition.

Arcwise connected. A point set S is **arcwise connected** if, for all $a, b \in S$, there exists a continuous curve lying entirely in S that connects a and b. That is, there is a continuous function $f : [0, 1] \to S$ with $f(0) = a$ and $f(1) = b$.

Capacitor-connected. A point set S is **capacitor-connected** if, for all $a, b \in S$ and all positive real numbers δ, there exists a sequence of points $a_0, \ldots, a_n \in S$ with $a_0 = a$, $a_n = b$, and $a_i a_{i+1} < \delta$ for $i = 0, \ldots, n-1$.

2 All unsplittable sets are capacitor-connected

Let S be an unsplittable point set, let $a \in S$, and let δ be a positive real number. Generate a sequence of point sets $A_0, A_1, \ldots$ as follows: $A_0 = \{a\}$, and for all $i \geq 0$,

$$A_{i+1} = S \cap \bigcup_{x \in A_i} Ball_\delta(x).$$

Let $A^* = \bigcup_{i \geq 0} A_i$.

Now, let

$$U = \bigcup_{x \in A^*} Ball_{\delta/2}(x) \quad \text{and} \quad V = \bigcup_{x \in S \setminus A^*} Ball_{\delta/2}(x).$$

Some observations about the sets U and V:

- U and V are both open because they are both unions of open balls.
- Together they cover S because $A^* \subseteq U$ and $S \setminus A^* \subseteq V$.

- They are disjoint. To prove this, suppose there were a point x in both sets. By the definition of U, there must be a point $x_U \in A^*$ with $xx_U < \delta/2$, and since $x_U \in A^*$, $x_U \in A_n$ for some n. By the definition of V, there must also be a point $x_V \in S \setminus A^*$ with $xx_V < \delta/2$. But

$$x_U x_V \leq xx_U + xx_V < \delta/2 + \delta/2 = \delta,$$

so $x_V \in Ball_\delta(x_U) \subseteq A_{n+1}$ by the definition of the A_i. But then $x_V \in A^*$, contradicting the fact that $x_V \in S \setminus A^*$.

- $U \cap S$ is nonempty (it contains a).

If $V \cap S$ were also nonempty, U and V would split S, but we know S is unsplittable, so V doesn't intersect S. This means $S \setminus A^* = \emptyset$, because if there existed $x \in S \setminus A^*$, we'd have $x \in V \cap S$. Thus $A^* = S$.

By the definition of A^*, any point $b \in S$ must be in A_n for some n. Generate a sequence of points $b_n, \ldots, b_0$ as follows: $b_n = b$, and $b_{i-1} \in A_{i-1}$ with $b_i b_{i-1} < \delta$ for $i = n, \ldots, 1$. A b_i must exist at every step because of the way the A_i were constructed, and $b_0 = a$ because a is the only point in A_0.

The points $a = b_0,\ b_1, \ldots,\ b_n = b$ form a sequence of hops in S from a to b of lengths less than δ. The foregoing argument can be used to create such a sequence for all $a, b \in S$ and all positive real numbers δ, so S is capacitor-connected. ■

3 All compact, capacitor-connected sets are unsplittable

Let S be a compact, capacitor-connected set split by sets T_1 and T_2; we'll prove the section title by reaching a contradiction.

Lemma. *Let O and C be open and compact point sets, respectively. If $O \supseteq C$, there exists a positive real number δ such that for all points $x \in C$ and $y \notin O$, $xy \geq \delta$.*

Proof: For $i = 0, 1, \ldots$, let O_i be the set of all points x such that $Ball_{2^{-i}}(x) \subseteq O$. Clearly $O_i \subseteq O_j$ when $i \leq j$. If x is any point in O, since O is open, there exists a positive real number r such that $Ball_r(x) \subseteq O$, so $x \in O_i$ where $i = max(0, \lceil -log_2 r \rceil)$; thus $\bigcup_i O_i = O$.

The O_i form an infinite set of open sets that together cover O and therefore C, so by the Heine-Borel Theorem, there also exists a finite set U of O_i whose members together cover C. Let n be the highest index of any of the sets O_i in U. Since $O_n \supseteq O_i$ for all $i \leq n$, $O_n = \bigcup U$ and therefore $O_n \supseteq C$.

Let $\delta = 2^{-n}$. Let x be any point in C. Since $C \subseteq O_n$, the definition of O_n tells us that $Ball_{2^{-n}}(x) \subseteq O$. Let y be any point not in O. Since $Ball_{2^{-n}}(x) \subseteq O$, $y \notin Ball_{2^{-n}}(x)$, so $xy \geq 2^{-n} = \delta$. □

Applying Lemma 1 with $O \mapsto T_1 \cup T_2$ and $C \mapsto S$, we find that any point x in S is always at least a fixed distance δ from any point y not in either T_i. I claim now that any point x in $S \cap T_1$ is further than δ from any point $y \notin T_1$; it is enough to prove this for $y \in T_2$.

Let $f : xy \to \mathbb{R}^n$ parametrize the line segment connecting x and y in the obvious fashion with $f(0) = x$ and $f(xy) = y$. Clearly $f(t_1)f(t_2) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, xy]$. Let P be the set of all $t \in [0, xy]$ for which $f(t) \in T_1$.

P is nonempty (it contains 0) and has an upper bound (xy), so by the completeness axiom, it has a supremum s. That is, $t \notin P$ for all $t > s$, and there exists no $s' < s$ for which $t \notin P$ for all $t > s'$. The second part implies that, for all $s' < s$, there exists $t \in (s', s]$ with $t \in P$. I now claim that $f(s)$ is a point in neither T_1 nor T_2.

If $f(s)$ were in T_1, then, since T_1 is open, there would exist a positive real number ϵ such that $Ball_\epsilon(f(s)) \subseteq T_1$. In particular, $f(s + \epsilon/2)$ would be in T_1, so $s + \epsilon/2 \in P$, contradicting the last paragraph. If $f(s)$ were in T_2, there would again exist a positive real number ϵ such that $Ball_\epsilon(f(s)) \subseteq T_2$. Setting $s' = s - \epsilon$, we find that there is a number $t \in (s', s]$ with $t \in P$, which means $f(t) \in T_1$. But $f(s)f(t) < \epsilon$, so $f(t) \in T_2$, contradicting the fact that T_1 and T_2 are disjoint.

We know that $x \in S$ and $f(s) \in \mathbb{R}^n \setminus (T_1 \cup T_2)$, so Lemma 1 tells us that $xf(s) \geq \delta$. $f(s)$ is strictly between x and y on a line segment, so $xy > xf(s) \geq \delta$, proving the claim that $xy \geq \delta$ for all $x \in S \cap T_1$ and $y \notin T_1$.

Since T_1 and T_2 split S, there exist points $a \in S \cap T_1$ and $b \in S \cap T_2$. Since S is capacitor-connected, there must exist points $a_0, \ldots, a_k \in S$ with $a_0 = a$, $a_k = b$, and $a_i a_{i+1} < \delta$ for $i = 0, \ldots, n-1$. Let j be the smallest index for which $a_j \in T_1$ but $a_{j+1} \notin T_1$; some such j must exist because $a_0 \in T_1$ and $a_k \notin T_1$. Applying the claim, we find that $a_j a_{j+1} > \delta$, contradicting the validity of the sequence of hops a_i. ■